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A gravitational constant and a cosmological constant in a spin-connection gravitational gauge field theory

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Abstract
A spin-connection gravitational gauge theory with a spin-affine connection as its fundamental dynamical variable is suggested in the framework of vierbein formulation. The functional integral approach to the interaction between complex scalar matter fields and a heavy intermediate coupling field is considered, where the Einstein field equation appears as a first-integral solution to the low-energy spin-connection gauge field equation of Yang–Mills type. The most intriguing characteristics of the present scheme include: (i) the gravitational constant originates from the low-energy propagator of the heavy coupling field that mediates the gravitation between the matter fields and the spin-connection gauge field; (ii) the large cosmological constant resulting from the quantum vacuum energy actually makes no gravitational contribution since the spin-connection gauge field equation is a third-order differential equation of the metric, and an integration constant of the first-integral solution serves as an effective cosmological constant that would cause the cosmic accelerated expansion. The present mechanism for interpreting the nonzero but small cosmological constant provides a new insight into the cosmological constant problem.

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1. Introduction

Though much experimental evidence has confirmed the validity and reliability of Einstein's general relativity (GR) in characterizing gravitation as a gauge interaction, describing the gravitational coupling of matters as well as interpreting cosmic evolution, physicists never stop attempting to establish new generalized gravity theories in order to unify gravity and other gauge fields [1–3], or to suggest modified gravity theories for interpreting new anomalous experimental phenomena [4–6]. These included, e.g., various versions of gravity with torsion (e.g., teleparallel gravity) [2, 7–10], modified Hilbert–Einstein theory (e.g., $1/R$-correction gravity) [3], coframe geometry and gravity [10, 11], and a variety of gauge approaches to gravitation [6] (including the Poincaré gauge theory [12, 13] and the metric-affine gauge theory of gravity [14]). It should be pointed out that most of these investigations were proposed based on the metric formalism, where the metric serves as a basic dynamical variable of the gravity theories. However, both the theoretical requirements and the recent observational results may have given a hint towards new field theories of gravitation with new dynamical variables. Take the gravitational gauge theory for example; we may need a proper fundamental dynamical variable for reformulating the gravity as a Yang–Mills type. The connection and the dynamical variable in the Yang–Mills gauge theory are the same quantity, while in GR, the connection (Levi–Civita connection) and the dynamical variable (metric) are not the same quantity, and then the field equation in general relativity is not a Yang–Mills type equation. In other words, the reason why the local Lorentz symmetry in the formulation of both the metric and Levi–Civita connections does not allow us to describe the gravitation as a Yang–Mills type gauge interaction, is because these two gauge interactions are formulated in different languages: specifically, GR is constructed using the Levi–Civita connection (in terms of the metric), while the Yang–Mills gauge field is described using a non-Abelian affine connection, which can actually be expressed in terms of the so-called 'Yang–Mills vielbein'. Likewise, there may exist a fundamental dynamical variable that can also be constructed in terms of the *vielbein* (in four dimensions we would call it a *vierbein*). This is one of the topics addressed in this paper. On the other hand, some recent astrophysical observations (e.g., Type Ia supernova observations [15]) showed that the large scale mean pressure of our present universe is negative suggesting a positive but small cosmological constant, and that the universe is therefore presently undergoing an accelerated expansion [15]. This means that we need a proper mechanism to interpret the nonzero but small cosmological constant. Whereas, the theoretically predicted cosmological constant that results from e.g. quantum vacuum energy is almost divergent (at least larger than the observed cosmological constant value by more than 120 orders of magnitude). We believe that, in order to resolve the present problem, we should elucidate the physical meanings of the cosmological constant from other possible aspects and new insights, and then interpret the cosmological constant problem by using new dynamical equations with new dynamical variables of gravitation (the new dynamics should certainly be reduced to the Einstein gravity under certain conditions). One such dynamics is the Yang–Mills type gravity theory, where the spin-affine connection (or the spin connection, for brevity) can serve as a fundamental dynamical variable for the gravitational gauge field.

The gravitational Lagrangian densities of the present Yang–Mills type gravitational field are the squares of the curvatures (including the Riemannian curvature and the scalar curvature). Here, the gravitational gauge field curvature (e.g., the Riemannian curvature in the vierbein formulation) is of the form of Yang–Mills type, which can be expressed explicitly in terms of the spin-affine connection. In a word, the present formalism of gauge field theory of Yang–Mills type for the gravitational interaction preserves a spin-connection local Lorentz-group gauge invariance. We will show that this scheme would be a new way to obtain the Einstein field equation of gravitation. Obviously, such a dynamics differs from the way based on the Hilbert–Einstein action, where the metric serves as the dynamical variable.

The most remarkable results of the present scheme include: (i) the Einstein field equation appears as a first-integral solution to the low-energy gravitational gauge field equation of spin-affine connection. In this sense, the Einstein gravity theory could be viewed as a low-energy phenomenological (effective) field theory of the present spin-connection gauge field theory; (ii) we suggest a scheme of matter–gravity interaction (mediated by a heavy intermediate coupling field) to formulate a theory of gravitation with a dimensionless fundamental coupling constant, and the Newtonian gravitational constant results from the low-energy Green function of the heavy coupling field whose contribution has been integrated in the vacuum–vacuum transition amplitude of the path integral approach. Therefore, the physical essence of the Newtonian gravitational constant is the low-energy propagator of the heavy intermediate coupling field (proportional to the mass square of the heavy coupling field); (iii) the cosmological constant due to quantum vacuum energy makes no gravitational contribution since the spin-connection gravitational field equation is a third-order differential equation of metric (the gravitation of the almost divergent cosmological constant caused by the quantum vacuum fluctuation is one of the puzzles in the cosmological constant problem. But now in the prescription of spin-connection gauge field theory, this puzzle could be removed). Besides, an equivalent cosmological constant that appears as an integration constant of the first-integral solution can naturally interpret the observed cosmological constant value that is close to the critical density of the universe.

This paper is organized as follows: we shall first construct a gravitational Lagrangian density by using the local Lorentz-group gauge symmetry, and obtain the variation of the gravitational action, where the spin connection is a dynamical variable. Then we introduce a heavy intermediate coupling field that mediates the gravitational interaction between the scalar matter field and the spin-connection gauge field, use the path integral approach to the matter and heavy intermediate coupling field, and derive a low-energy effective action of the matter field. In the low-energy case, the Einstein field equation can therefore be obtained via the variational principle from the low-energy effective action (where the heavy intermediate coupling field has already been integrated in the vacuum–vacuum transition amplitude). The present formalism of spin-connection gravitational gauge field theory provides us with new insights into the essence and the physical origins of both the gravitational constant and the cosmological constant.

2. Spin-connection gauge field theory and gravitational Lagrangian

As is well known, in the formulation of metric and Levi–Civita connections, the dynamical variable of gravitational field is the metric. We would, however, choose the spin connection as a fundamental dynamical variable for the gravitational field, and it will be shown that GR (in the formulation of vierbein and spin connection) would then become a Yang–Mills type gauge field theory. The Lagrangian formalism of spin-connection gauge theory of gravitation (with the Local Lorentz-group gauge invariance) will be suggested, where the candidates for the gravitational Lagrangian density are the local gauge-group invariant quadratic in curvatures.

In order to suggest a Yang–Mills type field equation of gravitation, we should first reformulate the Riemannian geometry and torsionless gravity using the formulation of vierbein, where the vierbein fields satisfy the relations $g_{\mu\nu} = \vartheta_{\mu}{}^{r}\vartheta_{\nu r}$ and $\delta_{\mu}{}^{\nu} = \vartheta_{\mu}{}^{r}\vartheta^{\nu}{}_{r}$. Here, the Greek and Latin indices denote the Einstein local coordinate indices and the spacetime indices (Lorentz coordinate indices) of the local inertial frame, respectively. In this paper, we choose the metric sign convention $(+---)$. In the Riemannian geometry (curvature-only torsion-free theory), the Levi–Civita covariant derivatives of the vierbeins are defined through $\nabla_{\lambda}\vartheta_{\nu r} = \partial_{\lambda}\vartheta_{\nu r} - \vartheta_{\sigma r}\Gamma^{\sigma}{}_{\lambda\nu}$ and $\nabla_{\lambda}\vartheta^{\nu}{}_{r} = \partial_{\lambda}\vartheta^{\nu}{}_{r} + \Gamma^{\nu}{}_{\lambda\sigma}\vartheta^{\sigma}{}_{r}$. Then it follows that the Levi–Civita affine connection can be expressed in terms of the vierbein fields. Take the Christoffel symbol $\Gamma^{\mu}{}_{\lambda\nu}$ for example, it can be written as $\Gamma^{\mu}{}_{\lambda\nu} = \vartheta^{\mu r}\partial_{\lambda}\vartheta_{\nu r} + S^{\mu}{}_{\lambda\nu}$, where $S^{\mu}{}_{\lambda\nu} = \vartheta_{\nu r}\nabla_{\lambda}\vartheta^{\mu r}$. Here, ∇_{λ} denotes the Levi–Civita covariant derivative.

Now we rewrite the Riemannian curvature tensor (using the vierbein formulation) in terms of the spin-connection gauge field. By using the definition of Levi–Civita covariant derivative, the second-order covariant derivative of the vierbein $\vartheta_\tau{}^r$ yields a relation $\vartheta_\tau{}^r{}_{;\mu;\nu} - \vartheta_\tau{}^r{}_{;\nu;\mu} = \vartheta_\beta{}^r R^\beta{}_{\tau\mu\nu}$. This leads to

$$\vartheta_\beta{}^r R^\beta{}_{\tau\mu\nu}\vartheta^{\tau s} = \left(\vartheta_\tau{}^r{}_{;\mu;\nu} - \vartheta_\tau{}^r{}_{;\nu;\mu}\right)\vartheta^{\tau s}. \tag{1}$$

We define a gauge field tensor $(\Omega_{\mu\nu})^{rs} \equiv \vartheta_\beta{}^r R^\beta{}_{\tau\mu\nu}\vartheta^{\tau s}$, and then we have

$$(\Omega_{\mu\nu})^{rs} = \left(\vartheta_\tau{}^r{}_{;\mu;\nu} - \vartheta_\tau{}^r{}_{;\nu;\mu}\right)\vartheta^{\tau s}. \tag{2}$$

In the vierbein formulation of torsion-free gravity, one can show that the gauge field tensor can be expressed as

$$(\Omega_{\mu\nu})^{rs} = \frac{1}{\mathrm{i}}(\nabla_\mu \check{S}_\nu - \nabla_\nu \check{S}_\mu - \mathrm{i}[\check{S}_\mu, \check{S}_\nu])^{rs}, \tag{3}$$

or $\Omega_{\mu\nu} = (\nabla_\mu \check{S}_\nu - \nabla_\nu \check{S}_\mu - \mathrm{i}[\check{S}_\mu, \check{S}_\nu])/\mathrm{i}$. Here, the Hermitian spin-affine connection (the Latin indices r, s are viewed as the matrix indices) is defined as $(\check{S}_\mu)^{rs} \equiv \mathrm{i}S^r{}_\mu{}^s = \mathrm{i}\vartheta_\alpha{}^r S^\alpha{}_\mu{}^\beta \vartheta_\beta{}^s$. It follows that a Hermitian spin-connection gauge field strength (curvature) can be defined as $\check{\Omega}_{\mu\nu} = \mathrm{i}\Omega_{\mu\nu}$. Thus, one can have $\check{\Omega}_{\mu\nu} = \nabla_\mu \check{S}_\nu - \nabla_\nu \check{S}_\mu - \mathrm{i}[\check{S}_\mu, \check{S}_\nu]$ whose component (matrix element) is given by

$$(\check{\Omega}_{\mu\nu})^{rs} = (\nabla_\mu \check{S}_\nu - \nabla_\nu \check{S}_\mu - \mathrm{i}[\check{S}_\mu, \check{S}_\nu])^{rs}. \tag{4}$$

This is a spin-connection non-Abelian gauge field strength.

We have so far focused on establishing the important relations in the vierbein formulation of gravitation. Now we are in a position to construct the Lagrangian densities for the gravitational field (spin-connection gravitational gauge field) by applying the gauge approach to the gravitational interactions.

In the tensor representation of the Lorentz group in four-dimensional spacetime, there are six group generators $(\mathcal{J}_{pq})^{rs}$, where the superscripts r, s are the matrix indices, and the subscripts (pq) are considered to be the number index of the six group generators, i.e., (pq) can be taken to be (01), (02), (03), (12), (13) and (23). In the Yang–Mills field theory, both the gauge field tensor and the gauge potential can be rewritten as the linear combinations of the gauge group generators $(T^i)^{ab}$, that is, we can have $F_{\mu\nu}{}^{ab} = F^i_{\mu\nu}(T^i)^{ab}$ and $A_\mu{}^{ab} = A^i_\mu(T^i)^{ab}$, where a, b denote the matrix indices of the group generators, and i is the number index of the group generators. Likewise, in the spin-connection gauge field theory of gravitation, both the gravitational gauge field tensor $(\check{\Omega}_{\mu\nu})^{rs}$ and the spin connection (dynamical variable) $(\check{S}_\mu)^{rs}$ can be expressed in terms of the Lorentz-group generators $(\mathcal{J}_{pq})^{rs}$ in the tensor representation, i.e.,

$$(\check{\Omega}_{\mu\nu})^{rs} = \frac{1}{2\mathrm{i}}(\check{\Omega}_{\mu\nu})^{pq}(\mathcal{J}_{pq})^{rs}, \qquad (\check{S}_\mu)^{rs} = \frac{1}{2\mathrm{i}}(\check{S}_\mu)^{pq}(\mathcal{J}_{pq})^{rs}. \tag{5}$$

This means that the coefficient $(\check{\Omega}_{\mu\nu})^{pq}/\mathrm{i}$ can be used to construct the gravitational Lagrangian density. According to the Yang–Mills field theory, the Lagrangian density, $\mathcal{L}$, and the action, S, must be the local gauge-group invariants quadratic in curvature, and the Lagrangian density is $-(1/4)F^i_{\mu\nu}F^{i\mu\nu}$. Following this rule, the possible candidate for the gravitational Lagrangian density is $-(1/4)(\check{\Omega}_{\mu\nu}/\mathrm{i})^{pq}(\check{\Omega}^{\mu\nu}/\mathrm{i})_{pq}$. But it should be pointed out that $(\check{S}^\mu)_{pq}$ and $(\check{S}^\mu)_{qp}$ actually correspond to the same spin-connection gauge field simply because of the antisymmetry in p and q, i.e., $(\check{S}^\mu)_{pq} = -(\check{S}^\mu)_{qp}$. Therefore, $(\check{\Omega}^{\mu\nu})_{pq}$ and $(\check{\Omega}^{\mu\nu})_{qp}$ are in fact the same gauge field strength. Thus, in the candidate $-(1/4)(\check{\Omega}_{\mu\nu}/\mathrm{i})^{pq}(\check{\Omega}^{\mu\nu}/\mathrm{i})_{pq}$ for the gravitational Lagrangian density, the contribution of the same gauge field strength has been repeated in the summation over p, q. Thus, in order to avoid the double contributions of the same gauge field strength, the gravitational Lagrangian density should be written as $-(1/8)(\check{\Omega}_{\mu\nu}/\mathrm{i})^{pq}(\check{\Omega}^{\mu\nu}/\mathrm{i})_{pq}$, which becomes $-(1/8)(\check{\Omega}_{\mu\nu})^{pq}(\check{\Omega}^{\mu\nu})_{qp}$. Now we consider one of such gauge invariants that are quadratic in curvature

$$S_{\mathrm{g}}^{(1)} = \int \sqrt{-g}\mathcal{L}_{\mathrm{g}}^{(1)}\,\mathrm{d}^4x, \qquad \mathcal{L}_{\mathrm{g}}^{(1)} = -\frac{1}{8}(\check{\Omega}_{\mu\nu})^{pq}(\check{\Omega}^{\mu\nu})_{qp}, \tag{6}$$

where the curvature tensor (gauge field strength) takes the form

$$(\check{\Omega}_{\mu\nu})^{pq} = \left(\frac{\partial \check{S}_\nu}{\partial x^\mu} - \frac{\partial \check{S}_\mu}{\partial x^\nu} - \mathrm{i}[\check{S}_\mu, \check{S}_\nu]\right)^{pq}. \tag{7}$$

In order to derive the gauge field equation, we should first calculate the variation of the gravitational Lagrangian density with respect to the spin connection $(\check{S}_\mu)^{pq}$:

$$\begin{aligned}\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{g}}^{(1)}\right) &= -\frac{1}{2}\sqrt{-g}(\check{\Omega}^{\mu\nu})_{qp}\left(\frac{\partial\delta\check{S}_\nu}{\partial x^\mu} - \mathrm{i}\delta(\check{S}_\mu\check{S}_\nu)\right)^{pq}\\ &= \frac{1}{2}\sqrt{-g}\mathcal{D}_\mu(\check{\Omega}^{\mu\nu})_{qp}\delta(\check{S}_\nu)^{pq} + \mathrm{S.T.},\end{aligned} \tag{8}$$

where S.T. denotes the surface term (total divergence term), which leads to a surface integral, using the four-dimensional version of Gauss' theorem. The second local Lorentz gauge-group invariant that is quadratic in curvature (scalar curvature) is of the form

$$S_{\mathrm{g}}^{(2)} = \int \sqrt{-g}\mathcal{L}_{\mathrm{g}}^{(2)}\,\mathrm{d}^4x, \qquad \mathcal{L}_{\mathrm{g}}^{(2)} = -\frac{1}{8}\left(\vartheta^\mu{}_r(\check{\Omega}_{\mu\nu})^{rs}\vartheta^\nu{}_s\right)^2, \tag{9}$$

where $\vartheta^\mu{}_r(\check{\Omega}_{\mu\nu})^{rs}\vartheta^\nu{}_s = \mathrm{i}\vartheta^\mu{}_r(\vartheta^{\alpha r}R_{\alpha\beta\mu\nu}\vartheta^{\beta s})\vartheta^\nu{}_s$ (this scalar equals $\mathrm{i}R$). It should be noted that there is not such a counterpart (or analogue) in the Lagrangian of the conventional Yang–Mills gauge theory. However, it deserves consideration since it is really a gauge invariant quantity quadratic in curvature. The variation of the second local Lorentz gauge-group invariant for the gravitational Lagrangian density is given by

$$\begin{aligned}\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{g}}^{(2)}\right) &= -\frac{1}{4}\mathrm{i}\sqrt{-g}R\left(\vartheta^\mu{}_r\vartheta^\nu{}_s\right)\delta\left(\frac{\partial \check{S}_\nu}{\partial x^\mu} - \frac{\partial \check{S}_\mu}{\partial x^\nu} - \mathrm{i}[\check{S}_\mu, \check{S}_\nu]\right)^{rs}\\ &= -\frac{1}{4}\mathrm{i}\sqrt{-g}R\left(\vartheta^\mu{}_r\vartheta^\nu{}_s - \vartheta^\nu{}_r\vartheta^\mu{}_s\right)\left[\frac{\partial\delta(\check{S}_\nu)^{rs}}{\partial x^\mu} - \mathrm{i}\delta((\check{S}_\mu)^r{}_t(\check{S}_\nu)^{ts})\right]\\ &= \frac{1}{2}(C+D).\end{aligned} \tag{10}$$

Here, the term C is

$$\begin{aligned}C &= -\frac{1}{2}\mathrm{i}\sqrt{-g}R\left(\vartheta^\mu{}_r\vartheta^\nu{}_s - \vartheta^\nu{}_r\vartheta^\mu{}_s\right)\frac{\partial\delta(\check{S}_\nu)^{rs}}{\partial x^\mu}\\ &= \mathrm{i}\sqrt{-g}\nabla_\mu\left[\frac{1}{2}R\left(\vartheta^\mu{}_r\vartheta^\nu{}_s - \vartheta^\nu{}_r\vartheta^\mu{}_s\right)\right]\delta(\check{S}_\nu)^{rs} + \mathrm{S.T.}\\ &= \mathrm{i}\frac{1}{2}\sqrt{-g}(\partial_\mu R)\left(\vartheta^\mu{}_r\vartheta^\nu{}_s - \vartheta^\nu{}_r\vartheta^\mu{}_s\right)\delta(\check{S}_\nu)^{rs}\\ &\quad + \sqrt{-g}R\left(\vartheta^\mu{}_t\vartheta^\nu{}_s - \vartheta^\nu{}_t\vartheta^\mu{}_s\right)(\check{S}_\mu)^t{}_r\delta(\check{S}_\nu)^{rs} + \mathrm{S.T.},\end{aligned} \tag{11}$$

where the relations $\mathcal{D}_\mu\vartheta^\mu{}_r = \nabla_\mu\vartheta^\mu{}_r - \mathrm{i}(\check{S}_\mu)_r{}^t\vartheta^\mu{}_t = 0, \mathcal{D}_\mu\vartheta^\nu{}_s = \nabla_\mu\vartheta^\nu{}_s - \mathrm{i}(\check{S}_\mu)_s{}^t\vartheta^\nu{}_t = 0$ and the property of antisymmetry of $\vartheta^\alpha{}_s\vartheta^\beta{}_r - \vartheta^\alpha{}_r\vartheta^\beta{}_s$ in indices r, s have been applied. The term D in (10) is given by

$$\begin{aligned}D &= -\tfrac{1}{2}\sqrt{-g}R\left(\vartheta^\mu{}_r\vartheta^\nu{}_s - \vartheta^\nu{}_r\vartheta^\mu{}_s\right)\delta\left((\check{S}_\mu)^r{}_t(\check{S}_\nu)^{ts}\right)\\ &= -\sqrt{-g}R\left(\vartheta^\mu{}_t\vartheta^\nu{}_s - \vartheta^\nu{}_t\vartheta^\mu{}_s\right)(\check{S}_\mu)^t{}_r\delta(\check{S}_\nu)^{rs}.\end{aligned} \tag{12}$$

Obviously, D and the second term of the result in (11) exactly cancel, and the only retained terms are the first term of the result in (11) and the surface term. Therefore, one can arrive at

$$\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{g}}^{(2)}\right) = \mathrm{i}\frac{1}{4}\sqrt{-g}(\partial_\mu R)\left(\vartheta^\mu{}_r\vartheta^\nu{}_s - \vartheta^\nu{}_r\vartheta^\mu{}_s\right)\delta(\check{S}_\nu)^{rs} + \mathrm{S.T.}$$
$$= \frac{\mathrm{i}}{2}\sqrt{-g}\mathcal{D}_\mu\left[\frac{1}{2}R\left(\vartheta^\mu{}_r\vartheta^\nu{}_s - \vartheta^\nu{}_r\vartheta^\mu{}_s\right)\right]\delta(\check{S}_\nu)^{rs} + \mathrm{S.T.}, \qquad (13)$$

where $\mathcal{D}_\mu e_r{}^\mu = 0$ and $\mathcal{D}_\mu e_s{}^\nu = 0$ have been substituted.

Hence, the variation of the two invariants constructed for the gravitational action is

$$\delta\left(S_{\mathrm{g}}^{(1)} + S_{\mathrm{g}}^{(2)}\right) = \frac{1}{2}\int\sqrt{-g}\left\{\mathcal{D}_\mu(\check{\Omega}^{\mu\nu})_{sr} + \mathrm{i}\mathcal{D}_\mu\left[\frac{1}{2}R\left(\vartheta^\mu{}_r\vartheta^\nu{}_s - \vartheta^\nu{}_r\vartheta^\mu{}_s\right)\right]\right\}\delta(\check{S}_\nu)^{rs}\,\mathrm{d}^4x. \qquad (14)$$

If we define a quantity that is antisymmetric in both Greek and Latin indices as follows:

$$(\check{Y}^{\mu\nu})_{qp} = \frac{\mathrm{i}}{2}\left(\vartheta^\mu{}_q\vartheta^\nu{}_p - \vartheta^\mu{}_p\vartheta^\nu{}_q\right)R, \qquad (15)$$

then the variation (14) becomes

$$\delta\left(S_{\mathrm{g}}^{(1)} + S_{\mathrm{g}}^{(2)}\right) = \frac{1}{2}\int\sqrt{-g}\{\mathcal{D}_\mu(\check{\Omega}^{\mu\nu})_{qp} - \mathcal{D}_\mu(\check{Y}^{\mu\nu})_{qp}\}\delta(\check{S}_\nu)^{pq}\,\mathrm{d}^4x. \qquad (16)$$

We shall discuss the gravitational action $S_{\mathrm{g}} = S_{\mathrm{g}}^{(1)} + S_{\mathrm{g}}^{(2)}$: (i) obviously, it is a higher derivative gravity of metric (but quadratic in both spin connection and curvature). In the literature, there were many references which considered the four-derivative gravity (where the dynamical variable is the metric, however) [16]. But in our Yang–Mills formulation of gravity, the dynamical variable is the spin connection instead; (ii) there is a third candidate that is also quadratic in curvature, i.e., $g^{\mu\nu}\vartheta^{\beta t}(\check{\Omega}_{\nu\beta})_{tr}(\check{\Omega}_{\mu\alpha})^{rs}\vartheta^\alpha{}_s$, for constructing the Lagrangian and action. This term can be expressed in terms of the other two candidates ($\mathcal{L}_{\mathrm{g}}^{(1)}$ and $\mathcal{L}_{\mathrm{g}}^{(2)}$) because of the Gauss–Bonnet relation, so that we shall not take it into consideration in this paper (it can be demonstrated that the spin-connection gauge theory based on the gravitational action $S_{\mathrm{g}}^{(1)} + S_{\mathrm{g}}^{(2)}$ is self-consistent for achieving the Einstein field equation).

So far, we have focused on deriving the variation of the gravitational action whose fundamental dynamical variable is the spin-affine connection. We next show that the Einstein tensor will appear in the variation of $S_{\mathrm{g}}^{(1)} + S_{\mathrm{g}}^{(2)}$. With the help of $(\check{\Omega}^{\mu\nu})_{qp} = \mathrm{i}\vartheta^\alpha{}_q R_{\alpha\beta}{}^{\mu\nu}\vartheta^\beta{}_p$, one can have

$$\mathcal{D}_\mu(\check{\Omega}^{\mu\nu})_{qp} = \mathrm{i}\vartheta^\alpha{}_q\left(\nabla_\mu R_{\alpha\beta}{}^{\mu\nu}\right)\vartheta^\beta{}_p = \mathrm{i}\vartheta^\alpha{}_q\left(\nabla_\alpha R_\beta{}^\nu - \nabla_\beta R_\alpha{}^\nu\right)\vartheta^\beta{}_p. \qquad (17)$$

By using the definition of $(\check{Y}^{\mu\nu})_{qp}$ in (15), one can obtain

$$-\mathcal{D}_\mu(\check{Y}^{\mu\nu})_{qp}\delta(\check{S}_\nu)^{pq} = -\mathrm{i}\tfrac{1}{2}\left[\nabla_\alpha\left(g_\beta{}^\nu R\right) - \nabla_\beta\left(g_\alpha{}^\nu R\right)\right]\vartheta^\beta{}_p\vartheta^\alpha{}_q\delta(\check{S}_\nu)^{pq}. \qquad (18)$$

By using the results of (17) and (18), we find that the variation of the purely gravitational Lagrangian density appearing in (16) is given by

$$\{\mathcal{D}_\mu(\check{\Omega}^{\mu\nu})_{qp} - \mathcal{D}_\mu(\check{Y}^{\mu\nu})_{qp}\}\delta(\check{S}_\nu)^{pq}$$
$$= \mathrm{i}\vartheta^\alpha{}_q\left[\nabla_\alpha\left(R_\beta{}^\nu - \tfrac{1}{2}g_\beta{}^\nu R\right) - \nabla_\beta\left(R_\alpha{}^\nu - \tfrac{1}{2}g_\alpha{}^\nu R\right)\right]\vartheta^\beta{}_p\delta(\check{S}_\nu)^{pq}. \qquad (19)$$

Thus, the variation in (16) can be rewritten as

$$\delta S_{\mathrm{g}} = \frac{\mathrm{i}}{2}\int\left(\nabla_\alpha G_\beta{}^\nu - \nabla_\beta G_\alpha{}^\nu\right)\vartheta^\alpha{}_q\vartheta^\beta{}_p\delta(\check{S}_\nu)^{pq}\sqrt{-g}\,\mathrm{d}^4x, \qquad (20)$$

where the Einstein tensor $G_\beta{}^\nu$, $G_\alpha{}^\nu$ has been derived in the variation of the purely gravitational action S_{g} with respect to the spin-affine connection $(\check{S}_\nu)^{pq}$. This is an alternative way to get the Einstein tensor, where the metric tensor is no longer a fundamental dynamical variable.

It should be noted that the vacuum field equation of gravitation obtained from variation (20) is given by a zero tensor $\nabla_\alpha G_\beta{}^\nu - \nabla_\beta G_\alpha{}^\nu$, and then the numbers of the equations are different between $\nabla_\alpha G_\beta{}^\nu - \nabla_\beta G_\alpha{}^\nu = 0$ and the vacuum Einstein equation $G_\beta{}^\nu = 0$ (thus, they are not equivalent). We point out that this problem is related close to the so-called 'Stephenson–Kilmister–Yang equation'. In the literature, Stephenson, Kilmister and Yang independently suggested a new gravitational field equation known as the Stephenson–Kilmister–Yang (SKY) equation, where the Christoffel symbol (Levi–Civita connection) serves as a non-Abelian gauge field [17–20]. In general, the source-free SKY field equation can be written as $\nabla_\mu R^\mu{}_{\nu\alpha\beta} = 0$, or equivalently in the form $\nabla_\alpha R_{\beta\nu} - \nabla_\beta R_{\alpha\nu} = 0$, which is also a third-order differential equation of metric. It can be readily verified that the Einstein vacuum field equation has already been involved in the SKY field equation. However, the SKY equation has some other new solutions, which were viewed as *unphysical* solutions [21]. It was suggested that the SKY equation should be supplemented by further restrictions on the class of allowable spacetime in order to rule out the so-called unphysical solutions [21] (e.g., the geometrically degenerate cases of conformally flatness and decomposability of spacetime involved in the SKY equation [19, 22], and the unphysical metrics that belong to these degenerate classes [21]). The static, spherically symmetric solution to the SKY equation showed that the solar experiments cannot yet distinguish between the SKY equation and the Einstein vacuum gravitational equation [18]. We can also obtain the source-free SKY field equation from variation (20). Though the SKY equation and the present spin-connection gauge field equation may have some 'unphysical' solutions, yet they have already contained all the solutions of the Einstein vacuum field equation. Thus, the spin-connection gauge field equation (as well as the SKY equation) is of physical interest, since such formalisms are more general than the Einstein field equation (i.e., the latter has already been involved in the former).

In this section, the gravitational Lagrangian density in the spin-connection formalism has been suggested by following the principle of gauge field theory of Yang–Mills type, i.e., the Lagrangian for the spin-connection gauge theory of gravitation based on both the gravitational gauge field strength (curvature) and the local Lorentz symmetry has been constructed. In the sections that follow, we shall suggest a formalism for treating the coupling of gravitational field to matter by applying the gauge approach, and then evaluate the variation of the action of the matter field involved in the spin-connection gravitational gauge interaction.

3. Functional integration for complex scalar fields and heavy coupling fields

It can be readily verified that the variation of the Lagrangian density, $\mathcal{L}_\varphi$, of a scalar matter field with respect to the spin connection vanishes, if $\mathcal{L}_\varphi$ is a bilinear form, such as $\mathcal{L}_\varphi = \mathcal{L}(\varphi, \partial_\mu\varphi)$ (in this paper, we are concerned with only the scalar matter field as the gravitational source). In other words, according to the variational principle, $\mathcal{L}_\varphi$ cannot provide the gravitational field equation with a source term (only for the vector and spinor fields can the Lagrangian densities give rise to the source terms that have close relation to the spin of matter fields). In order to obtain a source term related to the energy–momentum tensor of the matter field, a possible mechanism is that an extra coupling field, ϕ, would be needed in order to result in a nonzero variation for the coupling of the scalar matter field to the spin connection. Since there are no experimental evidences for the existence of this additional coupling field in low-energy gravitational interactions, the present coupling field (should such exist) might be very heavy, and only in very high-energy gravitational processes can it exhibit its existence (or presence). Thus, such a coupling field can also be referred to as 'heavy gravitational mediator' or 'heavy gravitation-mediating particle' (an intermediary particle that mediates the gravitational interaction between the matter field and the spin-connection gauge field).

We shall now present the Lagrangian densities of the matter field and the heavy intermediate coupling field, and then use the path integral approach in order to obtain a low-energy effective Lagrangian (for the matter field as well as its coupling to the gravity).

The Lagrangian densities of the matter field (say, the complex scalar fields φ^*, φ) and the heavy intermediate coupling fields ϕ^*, ϕ are given by

$$\begin{aligned}\mathcal{L}_\varphi &= \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi,\\ \mathcal{L}_\phi &= \partial_\mu\phi^*\partial^\mu\phi - m_G^2\phi^*\phi,\\ \mathcal{L}_{\varphi-\phi} &= \xi(\partial_\mu\phi^*\partial^\mu\varphi - m^2\phi^*\varphi) + \xi^*(\partial_\mu\phi\partial^\mu\varphi^* - m^2\phi\varphi^*)\\ &= \xi\phi^*J + \xi^*\phi J^* + \text{S.T.},\end{aligned} \tag{21}$$

where the interaction Lagrangian density $\mathcal{L}_{\varphi-\phi}$ is constructed for the φ–ϕ coupling. Here, $J = -(\nabla_\mu\partial^\mu\varphi + m^2\varphi)$ and $J^* = -(\nabla_\mu\partial^\mu\varphi^* + m^2\varphi^*)$. S.T. denotes the surface term (divergence term) for the present interaction Lagrangian density $\mathcal{L}_{\varphi-\phi}$. Apparently, ξ and its complex conjugate ξ^* are the *dimensionless* coupling constants. It should be emphasized that there are no coupling constants that have nonzero dimension.

The generating functional (the transition amplitude from vacuum to vacuum) for both the heavy intermediate coupling fields ϕ^*, ϕ and the matter fields φ^*, φ in the functional integral approach is defined as

$$Z^{(\phi\varphi)} = \int [D\varphi^*][D\varphi][D\phi^*][D\phi]\exp\left\{-\mathrm{i}\int\sqrt{-g}\,\mathrm{d}^4x[-\mathcal{L}_\phi - \xi\phi^*J - \xi^*\phi J^* - \mathcal{L}_\varphi]\right\}. \tag{22}$$

The appearance of $Z^{(\phi\varphi)}$ can be explicitly expressed via the transformations $\phi\to\phi+\phi_0$, $\phi^*\to\phi^*+\phi_0^*$. Using these transformations, one can rewrite the Lagrangian density of the heavy coupling fields, e.g., $-\mathcal{L}_\phi = \frac{1}{2}\phi^*\left(\Box+m_G^2\right)\phi + \frac{1}{2}\phi\left(\Box+m_G^2\right)\phi^* + \phi^*\left(\Box+m_G^2\right)\phi_0 + \phi\left(\Box+m_G^2\right)\phi_0^* + \frac{1}{2}\phi_0^*\left(\Box+m_G^2\right)\phi_0 + \frac{1}{2}\phi_0\left(\Box+m_G^2\right)\phi_0^* + \text{S.T.}$, and then the integrand in the exponential factor of (22) becomes

$$\begin{aligned}-\mathcal{L}_\phi - \xi\phi^*J - \xi^*\phi J^* \to{}& \tfrac{1}{2}\phi^*\left(\Box+m_G^2-\mathrm{i}\epsilon\right)\phi + \tfrac{1}{2}\phi\left(\Box+m_G^2+\mathrm{i}\epsilon\right)\phi^* + \text{S.T.}\\ &+\phi^*\left(\Box+m_G^2-\mathrm{i}\epsilon\right)\phi_0 + \tfrac{1}{2}\phi_0^*\left(\Box+m_G^2-\mathrm{i}\epsilon\right)\phi_0 - \xi\phi^*J - \xi\phi_0^*J\\ &+\phi\left(\Box+m_G^2+\mathrm{i}\epsilon\right)\phi_0^* + \tfrac{1}{2}\phi_0\left(\Box+m_G^2+\mathrm{i}\epsilon\right)\phi_0^* - \xi^*\phi J^* - \xi^*\phi_0 J^*.\end{aligned} \tag{23}$$

Here $\pm\mathrm{i}\epsilon$ terms are introduced in order to dictate the path of integration round the poles at the energy that obeys $k_\mu k^\mu = m_G^2$ (i.e., to ensure the vacuum-to-vacuum boundary conditions), and this makes the functional integration (22) well defined [23, 24]. Let us now define two equations for $\phi_0(x)$ and $\phi_0^*(x)$: $\left(\Box+m_G^2-\mathrm{i}\epsilon\right)\phi_0(x) = \xi J(x)$ and $\left(\Box+m_G^2+\mathrm{i}\epsilon\right)\phi_0^*(x) = \xi^*J^*(x)$. Then the solutions $\phi_0(x)$ and $\phi_0^*(x)$ to these two equations are given by

$$\begin{aligned}\phi_0(x) &= -\xi\int\Delta_\mathrm{F}(x-y)J(y)\sqrt{-g(y)}\,\mathrm{d}^4y,\\ \phi_0^*(x) &= -\xi^*\int\Delta_\mathrm{F}^*(x-y)J^*(y)\sqrt{-g(y)}\,\mathrm{d}^4y,\end{aligned} \tag{24}$$

where the Feynman propagators $\Delta_\mathrm{F}(x-y)$, $\Delta_\mathrm{F}^*(x-y)$ of the heavy coupling fields (complex fields) agree with

$$\begin{aligned}\left(\Box_x+m_G^2-\mathrm{i}\epsilon\right)\Delta_\mathrm{F}(x-y) &= -\delta^4(x-y),\\ \left(\Box_x+m_G^2+\mathrm{i}\epsilon\right)\Delta_\mathrm{F}^*(x-y) &= -\delta^4(x-y).\end{aligned} \tag{25}$$

Here, $\delta^4(x-y)$ function satisfies the relations: $\int \delta^4(x-y)\sqrt{-g(y)}\,\mathrm{d}^4y = 1$ and $\int J(y)\delta^4(x-y)\sqrt{-g(y)}\,\mathrm{d}^4y = J(x)$. Therefore, the result in equation (23) can be rewritten as

$$\begin{aligned}
&-\mathcal{L}_\phi - \xi\phi^* J - \xi^*\phi J^* \\
&\quad\to \frac{1}{2}\phi^*\left(\Box + m_G^2 - \mathrm{i}\epsilon\right)\phi + \frac{1}{2}\phi\left(\Box + m_G^2 + \mathrm{i}\epsilon\right)\phi^* - \frac{1}{2}\xi\phi_0^* J - \frac{1}{2}\xi^*\phi_0 J^* + \mathrm{S.T.} \\
&\quad= \frac{1}{2}\phi^*\left(\Box + m_G^2 - \mathrm{i}\epsilon\right)\phi + \frac{1}{2}\phi\left(\Box + m_G^2 + \mathrm{i}\epsilon\right)\phi^* + \mathrm{S.T.} \\
&\qquad + \frac{\xi^*\xi}{2}\int [J^*(x)\Delta_{\mathrm{F}}(x-y)J(y) + J(x)\Delta_{\mathrm{F}}^*(x-y)J^*(y)]\sqrt{-g(y)}\,\mathrm{d}^4y. \qquad (26)
\end{aligned}$$

The explicit expression in (26) can then be used to calculate the functional integration (22).

The generating functional $Z^{(\phi\varphi)}$ can be separable, i.e., $Z^{(\phi\varphi)} = Z^{(\phi)}Z^{(\varphi)}$. With the help of expression (26), one can obtain

$$\begin{aligned}
Z^{(\phi)} &= \int [D\phi^*][D\phi]\exp\left\{-\mathrm{i}\int\sqrt{-g(x)}\,\mathrm{d}^4x\left[\frac{1}{2}\phi^*\left(\Box + m_G^2 - \mathrm{i}\epsilon\right)\phi\right.\right. \\
&\quad \left.\left. + \frac{1}{2}\phi\left(\Box + m_G^2 + \mathrm{i}\epsilon\right)\phi^*\right]\right\}, \qquad (27) \\
Z^{(\varphi)} &= \int [D\varphi^*][D\varphi]\exp\left\{\mathrm{i}\int\sqrt{-g(x)}\,\mathrm{d}^4x[\mathcal{L}_\varphi(x) + \mathcal{L}_{\mathrm{int}}(x)]\right\},
\end{aligned}$$

where the Lagrangian density of the low-energy interaction mediated by the heavy coupling fields is given by

$$\mathcal{L}_{\mathrm{int}} = -\frac{\xi^*\xi}{2}\int [J^*(x)\Delta_{\mathrm{F}}(x-y)J(y) + J(x)\Delta_{\mathrm{F}}^*(x-y)J^*(y)]\sqrt{-g(y)}\,\mathrm{d}^4y. \qquad (28)$$

This is a nonlocal functional for the scalar matter fields (after integrating the heavy coupling fields ϕ^*, ϕ in the functional integration). At present, however, it is the low-energy case that is of much interest to us, since it can have a close relation to the Einstein field equation. For this reason, in what follows we shall consider the low-energy propagators of the heavy intermediate coupling fields ϕ^*, ϕ.

By using the general coordinate transformation $g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} g_{\mu\nu} \frac{\partial x^\nu}{\partial x'^\beta}$, one can obtain the metric determinant $g' = \left|\frac{\partial x}{\partial x'}\right|^2 g$, i.e., $\sqrt{-g'} = \left|\frac{\partial x}{\partial x'}\right|\sqrt{-g}$. We define the volume element in the phase space as follows: $\mathrm{d}^4x = \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3, \mathrm{d}^4k = \mathrm{d}k_0\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3$. Since $\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu, \mathrm{d}k'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\,\mathrm{d}k_\nu$, we can have $\sqrt{-g'}\,\mathrm{d}^4x' = \sqrt{-g}\,\mathrm{d}^4x$ and $\mathrm{d}^4k'/\sqrt{-g'} = \mathrm{d}^4k/\sqrt{-g}$, and the invariant phase-space volume element $\mathrm{d}^4k'\,\mathrm{d}^4x' = \mathrm{d}^4k\,\mathrm{d}^4x$. By using the Fourier transformation, we have $1 = (1/2\pi)^4\int\exp\left[-\mathrm{i}\int k_q(\mathrm{d}x^q - \mathrm{d}y^q)\right]\mathrm{d}^4k\,\mathrm{d}^4y = (1/2\pi)^4\int\exp\left[-\mathrm{i}\int k_\mu(\mathrm{d}x^\mu - \mathrm{d}y^\mu)\right]\mathrm{d}^4k\,\mathrm{d}^4y$, which can be rewritten as $1 = \int\delta^4(x-y)\sqrt{-g(y)}\,\mathrm{d}^4y$. Thus, the $\delta^4(x-y)$ function in the present scenario is given by

$$\delta^4(x-y) = \frac{1}{(2\pi)^4}\frac{1}{\sqrt{-g(y)}}\int\mathrm{d}^4k\exp\left[-\mathrm{i}\int k_\mu(\mathrm{d}x^\mu - \mathrm{d}y^\mu)\right]. \qquad (29)$$

Let us now look at the low-energy Taylor series expansion of $\Delta_{\mathrm{F}}(x-y)$. With the aid of equations (25) and (29), the Feynman propagator of the heavy coupling field ϕ is of the form

$$\Delta_{\mathrm{F}}(x-y) = \frac{1}{(2\pi)^4}\frac{1}{\sqrt{-g(y)}}\int\mathrm{d}^4k\frac{\exp\left[-\mathrm{i}\int k_\mu(\mathrm{d}x^\mu - \mathrm{d}y^\mu)\right]}{k^2 - m_G^2 + \mathrm{i}\epsilon}. \qquad (30)$$

Here, we have assumed that $k_\mu k^\mu$ is independent of the coordinate and $\nabla_\mu k^\mu = 0$, which are naturally the covariant generalizations of the results in the flat-spacetime quantum field theory.

In the case of low-energy gravitational interaction, $\Delta_{\mathrm{F}}(x-y)$ can become

$$\begin{aligned}\Delta_{\mathrm{F}}(x-y) &= -\frac{1}{m_G^2}\frac{1}{(2\pi)^4}\frac{1}{\sqrt{-g(y)}}\int \mathrm{d}^4k\frac{\exp\left[-\mathrm{i}\int k_\mu(\mathrm{d}x^\mu-\mathrm{d}y^\mu)\right]}{1-\frac{k^2+\mathrm{i}\epsilon}{m_G^2}}\\ &= -\frac{1}{m_G^2}\sum_{n=0}^{\infty}\frac{1}{(2\pi)^4}\frac{1}{\sqrt{-g(y)}}\int \mathrm{d}^4k\left(\frac{k_\nu k^\nu}{m_G^2}\right)^n\exp\left[-\mathrm{i}\int k_\mu(\mathrm{d}x^\mu-\mathrm{d}y^\mu)\right]. \qquad (31)\end{aligned}$$

Then, the appearance of equation (31) can be simplified, i.e., it can be rewritten as

$$\begin{aligned}\Delta_{\mathrm{F}}(x-y) &= -\frac{1}{m_G^2}\sum_{n=0}^{\infty}\left(\frac{-\Box_x}{m_G^2}\right)^n\frac{1}{(2\pi)^4}\frac{1}{\sqrt{-g(y)}}\int \mathrm{d}^4k\exp\left[-\mathrm{i}\int k_\mu(\mathrm{d}x^\mu-\mathrm{d}y^\mu)\right]\\ &= -\frac{1}{m_G^2}\left(1+\sum_{n=1}^{\infty}\mathcal{O}_n\right)\delta^4(x-y), \qquad (32)\end{aligned}$$

where the high-order term $\mathcal{O}_n = \left(-\Box_x/m_G^2\right)^n$. Under the condition of the low-energy interaction (below the energy scale $|m_G|$), the terms $\left(-\Box_x/m_G^2\right)^n$ can be ignored. Hence, the leading term of $\Delta_{\mathrm{F}}(x-y)$ is given by

$$\Delta_{\mathrm{F}}(x-y) \simeq -\frac{1}{m_G^2}\delta^4(x-y). \qquad (33)$$

We proceed to evaluate the effective interaction Lagrangian density. Substituting the leading term (33) of the propagator of the heavy intermediate coupling field into equation (28), one can obtain the interaction Lagrangian density (between the matter fields and the spin-connection gravitational gauge field) up to the first order

$$\mathcal{L}_{\mathrm{int}}(x) = \frac{\xi^*\xi}{m_G^2}J^*(x)J(x). \qquad (34)$$

It is, however, no longer a nonlocal functional. Since we have integrated the heavy intermediate coupling fields (ϕ^*, ϕ) in the vacuum–vacuum transition amplitude, as a result, such a kind of heavy intermediate fields lead to a J–J type Lagrangian with a coupling coefficient that is inversely proportional to the square of energy scale $|m_G|$ (the parameter $\xi^*\xi/m_G^2$ in (34) is expected to have a close relation to the Newtonian gravitational constant G). Apparently, the low-energy effective Lagrangian density $\mathcal{L}_{\mathrm{int}}$ is a modification to the Lagrangian density $\mathcal{L}_\varphi$ of the matter fields involved in the low-energy gravitational interaction.

4. The variation of the effective Lagrangian density

In this section, we shall concentrate our attention on the variational principle of the low-energy $\mathcal{L}_{\mathrm{int}}$. From the point of view of the spin-connection gauge field theory of gravitation, $\mathcal{L}_{\mathrm{int}}$ governs the interaction between the matter fields and the spin-connection gauge field (since the variation of $\mathcal{L}_{\mathrm{int}}$ with respect to the spin connection is nonzero). We shall now derive the nonzero variation of $\mathcal{L}_{\mathrm{int}}$. This will lead to the Einstein gravitational field equation, which is in fact a first-integral solution to the low-energy spin-connection gauge field equation.

In the expression for J^*J, there is a term $\nabla_\mu\partial^\mu\varphi^*\nabla_\nu\partial^\nu\varphi$ that deserves consideration (other terms, such as $m^2(\varphi^*\nabla_\mu\partial^\mu\varphi+\varphi\nabla_\mu\partial^\mu\varphi^*)$, have zero variations with respect to the spin-affine connection, and can be ignored). This term can be rewritten as $\nabla_\mu\partial^\mu\varphi^*\nabla_\nu\partial^\nu\varphi = R^{\mu\nu}\partial_\mu\varphi^*\partial_\nu\varphi+\nabla_\mu\partial_\nu\varphi^*\nabla^\mu\partial^\nu\varphi+\mathrm{S.T.}$, where we have used the relation $\nabla_\mu\partial_\nu\varphi = \nabla_\nu\partial_\mu\varphi$ (for the case of torsion-free gravity). For convenience, in what follows we focus on treating the following relation:

$$2\nabla_\mu\partial^\mu\varphi^*\nabla_\nu\partial^\nu\varphi = R^{\mu\nu}\pi_{\mu\nu}+2\nabla_\mu\partial_\nu\varphi^*\nabla^\mu\partial^\nu\varphi+\mathrm{S.T.}, \qquad (35)$$

where the symmetric tensor $\pi_{\mu\nu} = \partial_\mu\varphi^*\partial_\nu\varphi + \partial_\nu\varphi^*\partial_\mu\varphi$. In the vierbein formulation, the term $R^{\mu\nu}\pi_{\mu\nu}$ in relation (35) can be rewritten as

$$R^{\mu\nu}\pi_{\mu\nu} = (\check{\Omega}_{\mu\nu})^{pq}(\check{\mathcal{C}}^{\mu\nu})_{qp}, \tag{36}$$

where the source tensor $(\check{\mathcal{C}}^{\mu\nu})_{qp}$ of the present matter fields (complex scalar fields φ^*, φ) is defined by

$$(\check{\mathcal{C}}^{\mu\nu})_{qp} = \frac{\mathrm{i}}{4}\left[\left(\vartheta^\mu{}_q\pi^\nu{}_p - \vartheta^\nu{}_q\pi^\mu{}_p\right) - \left(\vartheta^\mu{}_p\pi^\nu{}_q - \vartheta^\nu{}_p\pi^\mu{}_q\right)\right]. \tag{37}$$

Then the variation of the first term on the right-handed side of equation (35) is given by

$$\begin{aligned}\delta(\sqrt{-g}R^{\mu\nu}\pi_{\mu\nu}) &= -2\sqrt{-g}\mathcal{D}_\mu(\check{\mathcal{C}}^{\mu\nu})_{qp}\delta(\check{S}_\nu)^{pq} + \text{S.T.}\\ &= \frac{\mathrm{i}}{2}\sqrt{-g}\left(\nabla_\beta\pi_\alpha{}^\nu - \nabla_\alpha\pi_\beta{}^\nu\right)\vartheta^\alpha{}_q\vartheta^\beta{}_p\delta(\check{S}_\nu)^{pq}\\ &\quad + \frac{\mathrm{i}}{2}\sqrt{-g}\left(g_\alpha{}^\nu\nabla_\mu\pi^\mu{}_\beta - g_\beta{}^\nu\nabla_\mu\pi^\mu{}_\alpha\right)\vartheta^\alpha{}_q\vartheta^\beta{}_p\delta(\check{S}_\nu)^{pq} + \text{S.T.},\end{aligned} \tag{38}$$

where the antisymmetry of $(\check{\mathcal{C}}^{\mu\nu})_{qp}$ in both μ, ν and q, p has been applied. It should be emphasized that here the spin connection is not involved in the source tensor $\pi^\mu{}_p, \pi^\mu{}_q, \pi^\nu{}_p$ and $\pi^\nu{}_q$ (i.e., the variation of these source tensors with respect to the spin connection is zero). This is true for the zero-spin particle and the macroscopic matter whose spin can be ignored. The variation of the second term on the right-handed side of equation (35) is $\delta(2\sqrt{-g}\mathcal{D}_\nu\partial^p\varphi^*\mathcal{D}^\nu\partial_p\varphi)$, where the spin-connection covariant derivatives are defined by: $\mathcal{D}_\nu\partial^p\varphi^* = \partial_\nu\partial^p\varphi^* - \mathrm{i}(\check{S}_\nu)^p{}_q\partial^q\varphi^*$ and $\mathcal{D}^\nu\partial_p\varphi = \partial^\nu\partial_p\varphi - \mathrm{i}(\check{S}^\nu)_p{}^q\partial_q\varphi$. Thus, we have

$$\begin{aligned}\delta(2\sqrt{-g}\mathcal{D}_\nu\partial^p\varphi^*\mathcal{D}^\nu\partial_p\varphi) &= 2\sqrt{-g}[\mathcal{D}^\nu\partial_p\varphi(-\mathrm{i}\partial_q\varphi^*) + \mathcal{D}^\nu\partial_p\varphi^*(-\mathrm{i}\partial_q\varphi)]\delta(\check{S}_\nu)^{pq}\\ &= -\mathrm{i}\sqrt{-g}[(\partial_\alpha\varphi\nabla^\nu\partial_\beta\varphi^* + \partial_\alpha\varphi^*\nabla^\nu\partial_\beta\varphi) - (\partial_\beta\varphi\nabla^\nu\partial_\alpha\varphi^* + \partial_\beta\varphi^*\nabla^\nu\partial_\alpha\varphi)]\vartheta^\alpha{}_q\vartheta^\beta{}_p\delta(\check{S}_\nu)^{pq}\\ &= -\mathrm{i}\sqrt{-g}[\nabla_\beta(\partial_\alpha\varphi^*\partial^\nu\varphi + \partial_\alpha\varphi\partial^\nu\varphi^*) - \nabla_\alpha(\partial_\beta\varphi^*\partial^\nu\varphi + \partial_\beta\varphi\partial^\nu\varphi^*)]\vartheta^\alpha{}_q\vartheta^\beta{}_p\delta(\check{S}_\nu)^{pq}\\ &= -\mathrm{i}\sqrt{-g}\left(\nabla_\beta\pi_\alpha{}^\nu - \nabla_\alpha\pi_\beta{}^\nu\right)\vartheta^\alpha{}_q\vartheta^\beta{}_p\delta(\check{S}_\nu)^{pq}.\end{aligned} \tag{39}$$

Therefore, based on equations (38) and (39), the variation of (35) takes the form

$$\begin{aligned}&\delta[\sqrt{-g}(R^{\mu\nu}\pi_{\mu\nu} + 2\nabla_\mu\partial_\nu\varphi^*\nabla^\mu\partial^\nu\varphi)]\\ &\quad = -\frac{\mathrm{i}}{2}\sqrt{-g}\left[\left(\nabla_\beta\pi_\alpha{}^\nu - \nabla_\alpha\pi_\beta{}^\nu\right) - \left(g_\alpha{}^\nu\nabla_\mu\pi^\mu{}_\beta - g_\beta{}^\nu\nabla_\mu\pi^\mu{}_\alpha\right)\right]\vartheta^\alpha{}_q\vartheta^\beta{}_p\delta(\check{S}_\nu)^{pq} + \text{S.T.}\\ &\quad = -\frac{\mathrm{i}}{2}\sqrt{-g}\left[\left(\nabla_\beta\tau_\alpha{}^\nu - \nabla_\alpha\tau_\beta{}^\nu\right) - \left(g_\alpha{}^\nu\nabla_\mu\tau^\mu{}_\beta - g_\beta{}^\nu\nabla_\mu\tau^\mu{}_\alpha\right)\right]\vartheta^\alpha{}_q\vartheta^\beta{}_p\delta(\check{S}_\nu)^{pq} + \text{S.T.},\end{aligned} \tag{40}$$

where $\tau_\alpha{}^\nu = \pi_\alpha{}^\nu - g_\alpha{}^\nu\mathcal{L}_\varphi$ and $\tau^\mu{}_\beta = \pi^\mu{}_\beta - g^\mu{}_\beta\mathcal{L}_\varphi$, which can be referred to as the 'quasi energy–momentum tensors' of the matter fields. It should be emphasized that the real energy–momentum tensors, such as $T_\alpha{}^\nu, T^\mu{}_\beta$, can be derived via the Noether theorem applied to $\mathcal{L}_\varphi + \mathcal{L}_{\text{int}}$. The quasi energy–momentum tensors $\tau_\alpha{}^\nu$ and $\tau^\mu{}_\beta$ are simply a part of the real energy–momentum tensors $T_\alpha{}^\nu$ and $T^\mu{}_\beta$, respectively.

We have so far focused on the variation of $\mathcal{L}_{\text{int}}$. It should be noted that the appearance of (40) can be simplified, if we take advantage of the property of the quasi energy–momentum tensor $\tau_\alpha{}^\nu$ (i.e., its covariant divergence vanishes). By the aid of $\mathcal{L}_\varphi = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi$ and $\mathcal{L}_{\text{int}} = \left(\xi^*\xi/m_G^2\right)(\nabla_\mu\partial^\mu\varphi^* + m^2\varphi^*)(\nabla_\nu\partial^\nu\varphi + m^2\varphi)$, one can obtain the equations of motion of the matter fields φ^*, φ (in the low-energy field theory of gravitation), e.g., $\nabla_\mu\partial^\mu\varphi + m^2\varphi = \left(\xi^*\xi/m_G^2\right)(\nabla_\mu\partial^\mu + m^2)(\nabla_\nu\partial^\nu\varphi + m^2\varphi)$. Obviously, the field equation $\nabla_\mu\partial^\mu\varphi + m^2\varphi = 0$ can satisfy this equation of motion. In the above, we have defined a quasi energy–momentum tensor $\tau^{\mu\nu} = \partial^\mu\varphi^*\partial^\nu\varphi + \partial^\mu\varphi\partial^\nu\varphi^* - g^{\mu\nu}\mathcal{L}_\varphi$ for the matter fields φ^*, φ. Thus, the covariant divergence of the present quasi energy–momentum tensor is $\nabla_\mu\tau^{\mu\nu} = (\nabla_\mu\partial^\mu\varphi^* + m^2\varphi^*)\partial^\nu\varphi + (\nabla_\mu\partial^\mu\varphi + m^2\varphi)\partial^\nu\varphi^* = 0$. Hence, the terms $\nabla_\mu\tau^\mu{}_\beta$ and

$\nabla_\mu \tau^\mu{}_\alpha$ in expression (40) vanish. Therefore, the simplified version for the variation of the low-energy Lagrangian density (34) takes the form

$$\delta(\sqrt{-g}\mathcal{L}_{\text{int}}) = \frac{\xi^*\xi}{2m_G^2}\left(-\frac{\mathrm{i}}{2}\right)\sqrt{-g}\left(\nabla_\beta \tau_\alpha{}^\nu - \nabla_\alpha \tau_\beta{}^\nu\right)\vartheta^\alpha{}_q \vartheta^\beta{}_p \delta(\check{S}_\nu)^{pq} + \text{S.T.}, \tag{41}$$

which, in combination with variation (20) of the gravitational action, can be used to derive the Einstein field equation (see the section that follows).

We next address the equation of motion of the heavy intermediate coupling field (in the case of high energy $|m_G|$ that would be close to the Planck energy) as well as the problem of its mass. According to the Lagrangian densities presented in equation (21), for the high-energy processes (e.g., the interaction energy scale is close to the Planck energy), the classical field equations of the matter field φ and the heavy coupling field ϕ are of the form: $\nabla_\mu \partial^\mu \varphi + m^2\varphi = -\xi^*(\nabla_\mu \partial^\mu \phi + m^2\phi)$, $\nabla_\mu \partial^\mu \phi + m_G^2\phi = -\xi(\nabla_\mu \partial^\mu \varphi + m^2\varphi)$. By substituting the first equation into the second one, one can show that the field equation of the heavy coupling field ϕ is $\nabla_\mu \partial^\mu \phi + m_G^2\phi = \xi^*\xi(\nabla_\mu \partial^\mu \phi + m^2\phi)$, which can be rewritten as

$$\nabla_\mu \partial^\mu \phi + \left(\frac{m_G^2 - \xi^*\xi m^2}{1-\xi^*\xi}\right)\phi = 0. \tag{42}$$

Here, the square of mass reads

$$M_G^2 = \frac{m_G^2 - \xi^*\xi m^2}{1-\xi^*\xi}. \tag{43}$$

In the following section, we shall present an expression for the gravitational constant G in terms of the mass, M_G, of the heavy intermediate coupling field, and indicate that the gravitational constant is in fact the low-energy propagator of the heavy coupling field.

5. The low-energy gravitational field equation and the gravitational constant

Now we are in a position to consider the total variations of matter and gravitational fields. It is possible, in fact, to demonstrate that the Einstein field equation has already been involved in the framework of spin-connection gravitational gauge field. With the help of equations (20) and (41), one can obtain

$$\delta(S_{\mathrm{g}} + S_{\text{eff}}[\varphi^*, \varphi]) = \frac{\mathrm{i}}{2}\int \sqrt{-g}\,\mathrm{d}^4x\left[\nabla_\alpha\left(G_\beta{}^\nu - \kappa\tau_\beta{}^\nu\right) - \nabla_\beta\left(G_\alpha{}^\nu - \kappa\tau_\alpha{}^\nu\right)\right]\vartheta^\alpha{}_q \vartheta^\beta{}_p \delta(\check{S}_\nu)^{pq}, \tag{44}$$

where the low-energy gravitational coupling constant $\kappa = -\xi^*\xi/2m_G^2$. It is necessary to discuss the relation between the low-energy gravitational coupling constant κ and the mass of the heavy intermediate coupling field. As the mass m of the matter fields is small, it can be ignored in equation (43). For the present, there are clearly no experimental evidences for the values of the dimensionless coupling constants ξ, ξ^*. If the gravitation dominates in the interactions of high-energy processes (it is believed that the gravitational interaction in high-energy processes close to the Planck energy scale would be strong. The gravitation in the low-energy case appears very weak simply because the intermediate coupling fields ϕ^*, ϕ are too heavy), as a tentative analysis, we can suppose that $\xi^*\xi \gg 1$, and then we can have

$$\kappa = \frac{1}{2M_G^2}. \tag{45}$$

Then, it is easy to obtain the relation between the mass of the heavy coupling field and the Newtonian gravitational constant G. It is apparent, for example, that one can get a first-integral solution

$$R_{\beta}{}^{\nu} - \frac{1}{2}g_{\beta}{}^{\nu}R - \kappa\tau_{\beta}{}^{\nu} = \Lambda g_{\beta}{}^{\nu}, \tag{46}$$

from variation (44). Here, Λ denotes an integration constant, which serves as an effective cosmological constant. Thus, we obtain the Einstein gravitational field equation as a first-integral solution of the spin-connection Yang–Mills type gauge field theory. According to the Einstein field equation, the low-energy coupling constant $\kappa = 8\pi G$ (in the unit of $\hbar = c = 1$), and then it follows from equation (45) that the gravitational constant G is given by

$$G = \frac{1}{16\pi M_G^2}, \tag{47}$$

i.e., the physical essence of the Newtonian gravitational constant has a close relation to the low-energy propagator of the heavy intermediate coupling field involved in gravitational processes. As is well known, the Planck mass $M_{\rm P} = \sqrt{1/G}$. Then it follows from relation (47) that the mass of the heavy intermediate coupling field ϕ is

$$M_G = \frac{M_{\rm P}}{4\sqrt{\pi}}. \tag{48}$$

This means that the mass of the heavy intermediate coupling field ϕ is close to the Planck mass.

In the above, we did not take account of the cosmological constant term in the Lagrangian density. If the cosmological constant term is taken into consideration, the spin-connection gauge theory would lead to a different route for looking at the cosmological constant problem from new aspects. As has been pointed out, $\tau_{\beta}{}^{\nu}$ and $\tau_{\alpha}{}^{\nu}$ in equations (40), (41) and (44) denote the quasi energy–momentum tensors of the matter field (complex scalar fields φ^{*}, φ). In a natural generalization to equations (41) and (44), the contribution of quantum-vacuum energy density ϱ can also be included, i.e., $\tau_{\beta}{}^{\nu} \rightarrow \tau_{\beta}{}^{\nu} + \varrho g_{\beta}{}^{\nu}$, $\tau_{\alpha}{}^{\nu} \rightarrow \tau_{\alpha}{}^{\nu} + \varrho g_{\alpha}{}^{\nu}$, where $\varrho = \lambda/\kappa$ (λ denotes the cosmological constant resulting from the quantum vacuum fluctuation). It is very interesting that the quantum-vacuum energy actually makes no contributions to gravitation, since the covariant derivatives of $\varrho g_{\beta}{}^{\nu}$ and $\varrho g_{\alpha}{}^{\nu}$ in (44) vanish. This, therefore, implies that the cosmological constant in the Yang–Mills type gravity theory, where the spin-affine connection becomes the dynamical variable of the local Lorentz-group gauge field, may have a different behaviour compared with that in GR. This would, unavoidably, change our understandings about the physical meanings and the roles of the cosmological constant in gravitational interactions.

6. Discussion of the cosmological constant in the framework of spin-connection gauge theory

Recently, a number of increasing evidences (e.g. measurements of the cosmic microwave background anisotropy, observations of the large-scale spacetime structure, and searches for type Ia supernovae [4]) have suggested that most of the energy density of the universe consists of matter that can be described by the cosmological constant term in the Einstein field equation. There may be two candidates for interpreting the origin of the cosmological constant, i.e., the quantum vacuum zero-point fluctuation energy and the dark energy (quintessence) [25, 26], which can exhibit negative pressure causing the cosmic expansion to accelerate. In an attempt to explain the accelerated expansion of the universe on the basis of GR and modern cosmology (the standard Friedmann–Robertson–Walker cosmology) [4], we meet, however, with difficulties arising from the cosmological constant problem [27]. There are some mysteries concerning the cosmological constant, for example, why does the very large (almost divergent) quantum vacuum zero-point energy density make no contributions to the observed cosmological constant, why is the observed cosmological constant nonzero but very small, and why does it take a value that is close to the critical energy density of the universe at present epoch? All these mysteries have puzzled physicists in the literature [27]. Take the vacuum fluctuation energy for example, the calculated quantum vacuum energy density is anomalously larger than the observed cosmological constant (close to the critical density) by even more than 120 orders of magnitude, i.e. the theoretical value of the cosmological constant (say, λ) caused by the quantum vacuum energy is surprisingly too large, while the experimental value of the cosmological constant (say, Λ) is surprisingly too small: specifically, λ resulting from the quantum vacuum contribution is proportional to $1/L_{\mathrm{P}}^2$ with L_{p} being the Planck length ($L_{\mathrm{P}} \simeq 10^{-35}$ m); the observed Λ is proportional to $1/L_{\mathrm{U}}^2$ with L_{U} being the cosmic length scale (the radius of the universe, $L_{\mathrm{U}} \simeq 10^{26}$ m). Thus, the ratio of the calculated λ to the observed Λ is: $\lambda/\Lambda \simeq (L_{\mathrm{U}}/L_{\mathrm{P}})^2$, which is more than 10^{120}. Although there would be an intrinsic bare cosmological constant that can eliminate λ, and the difference (the effective cosmological constant) would be expected to be close to the observed cosmological constant, yet we cannot accept such a surprising mechanism (fine tuning mechanism) in which the theoretical cosmological constant can be exactly cancelled and hence precisely adjusted by the bare cosmological constant by more than 120 orders of magnitude. Though a number of authors suggested many theories, including the scheme of suggesting new matter states such as the superconductivity of gravitomagnetic matter (a perfect fluid for exactly eliminating the divergent cosmological constant) [28–30], to interpret the nonzero but small cosmological constant [27], yet no satisfactory mechanisms have been widely accepted.

In the present spin-connection gauge field theory (in the vierbein formulation), the cosmological constant term appears as the spin-connection covariant derivative of the following antisymmetric tensor (antisymmetric in both the indices μ, ν and p, q)

$$(\lambda^{\mu\nu})_{qp} \equiv \frac{\mathrm{i}}{2}\lambda\left(\vartheta^{\mu}{}_{q}\vartheta^{\nu}{}_{p} - \vartheta^{\mu}{}_{p}\vartheta^{\nu}{}_{q}\right). \tag{49}$$

Obviously, the spin-connection covariant derivative of $(\lambda^{\mu\nu})_{qp}$ vanishes, i.e., $\mathcal{D}_{\mu}(\lambda^{\mu\nu})_{qp} \equiv 0$, and hence the cosmological term $(\lambda^{\mu\nu})_{qp}$ contributes nothing to the gravitation. This result can be interpreted in an alternative way, where the metric formulation is employed: specifically, the cosmological constant term (of the present spin-connection gravitational gauge theory) in the formulation of metric appears as the Levi–Civita covariant derivatives of $\varrho g_{\beta}{}^{\nu}$ and $\varrho g_{\alpha}{}^{\nu}$. We shall point out this scheme in more detail. It can be shown that the Lagrangian density of the cosmological constant term in the spin-connection gauge theory is proportional to the scalar curvature, e.g., $\mathcal{L}_{\lambda} = \lambda R/2$, which can be rewritten as $\mathcal{L}_{\lambda} = (\check{\Omega}_{\mu\nu})^{pq}(\lambda^{\mu\nu})_{qp}/2$. As the spin connection is $(\check{S}_{\nu})^{pq} = \mathrm{i}\vartheta_{\alpha}{}^{p}\nabla_{\nu}\vartheta^{\alpha q}$, the variation with respect to $(\check{S}_{\nu})^{pq}$ would be equivalent to the variation with respect to the covariant-derivative operator ∇_{ν}. There is a useful variational rule for a quantity such as $(\check{\Omega}_{\mu\nu})^{pq}(\chi^{\mu\nu})_{qp}$

$$\delta(\sqrt{-g}(\check{\Omega}_{\mu\nu})^{pq}(\chi^{\mu\nu})_{qp}) = -2\sqrt{-g}\mathcal{D}_{\mu}(\chi^{\mu\nu})_{qp}\delta(\check{S}_{\nu})^{pq} + \mathrm{S.T.}, \tag{50}$$

where the spin-connection gauge field strength $(\check{\Omega}_{\mu\nu})^{pq}$ is expressed in terms of the spin connection, while the tensor $(\chi^{\mu\nu})_{qp}$ does not contain the spin connection. If $(\chi^{\mu\nu})_{qp}$ is a function of the spin connection, then the above formula can be generalized by calculating the variation of $(\chi^{\mu\nu})_{qp}$. Thus, the variation of the Lagrangian density of the cosmological constant term is given by

$$\begin{aligned}\delta(\sqrt{-g}\mathcal{L}_{\lambda}) &= -\sqrt{-g}\mathcal{D}_{\mu}(\lambda^{\mu\nu})_{qp}\delta(\check{S}_{\nu})^{pq} + \mathrm{S.T.}\\ &= \frac{\mathrm{i}}{2}\sqrt{-g}\left[\nabla_{\beta}\left(\lambda g_{\alpha}{}^{\nu}\right) - \nabla_{\alpha}\left(\lambda g_{\beta}{}^{\nu}\right)\right]\vartheta^{\alpha}{}_{q}\vartheta^{\beta}{}_{p}\delta(\check{S}_{\nu})^{pq} + \mathrm{S.T.},\end{aligned} \tag{51}$$

where λ is taken to be $-\left(\xi^{*}\xi\big/2m_{G}^{2}\right)\varrho$ according to the relations $\kappa=-\xi^{*}\xi\big/2m_{G}^{2}$ and $\varrho=\lambda/\kappa$ that have been used in the preceding section. It follows from equations (44) and (51) that the gravitational field equation (containing the contribution of quantum vacuum energy) in the spin-connection formalism is of the form

$$\nabla_{\alpha}\left[G_{\beta}{}^{\nu}-\kappa\left(\tau_{\beta}{}^{\nu}+\varrho g_{\beta}{}^{\nu}\right)\right]-\nabla_{\beta}\left[G_{\alpha}{}^{\nu}-\kappa\left(\tau_{\alpha}{}^{\nu}+\varrho g_{\alpha}{}^{\nu}\right)\right]=0. \tag{52}$$

It is clearly seen that the quantum-vacuum energy density ϱ actually makes no contributions to the gravitation. This would be the reason for why the gravitational effect of the almost divergent vacuum energy density has so far never been detected in the present observational cosmology [4]. On the other hand, it should be emphasized that there appears an effective cosmological constant Λ in the first-integral solution (46) to the third-order differential equation (52) of the metric. We can interpret this problem alternatively in the vierbein formulation: from equation (44), the spin-connection gravitational field equation in the vierbein formulation is

$$\frac{1}{2}\mathcal{D}_{\mu}[(\check{\Omega}^{\mu\nu})_{qp}-(\check{Y}^{\mu\nu})_{qp}]=-\frac{\mathrm{i}}{2}\kappa\mathcal{D}_{\mu}\left[\left(\vartheta^{\mu}{}_{p}\tau^{\nu}{}_{q}-\vartheta^{\mu}{}_{q}\tau^{\nu}{}_{p}\right)+\left(\vartheta^{\nu}{}_{p}\tau^{\mu}{}_{q}-\vartheta^{\nu}{}_{q}\tau^{\mu}{}_{p}\right)\right]. \tag{53}$$

It is clearly seen that there is a spin-connection covariant divergence $\mathcal{D}_{\mu}\{\cdot\}^{\mu}$ on both left- and right-handed sides of equation (53). Therefore, an integration constant term such as

$$(\Lambda^{\mu\nu})_{qp}=\frac{\mathrm{i}}{2}\Lambda\left(\vartheta^{\mu}{}_{q}\vartheta^{\nu}{}_{p}-\vartheta^{\mu}{}_{p}\vartheta^{\nu}{}_{q}\right) \tag{54}$$

would appear in the first-integral solution, or equivalently, the term $\mathcal{D}_{\mu}(\Lambda^{\mu\nu})_{qp}$ (which equals zero) can be placed on the right-handed side in equation (53). Thus, the first-integral solution (46), i.e., the Einstein field equation that contains an equivalent cosmological term can be obtained from the present gravitational gauge field theory. Obviously, the parameter Λ, which is an integration constant, has nothing to do with the vacuum quantum fluctuation energy. According to the current data of cosmological observations, the value of the equivalent cosmological constant Λ is about $\lambda/10^{120}$, which is close to the critical density of the universe. We expect that the idea that we view the cosmological constant as an integration constant of the first-integral solution (46) may naturally interpret the observed cosmological constant value: specifically, the integration constant of the solutions (to the Yang–Mills type field equation presented in this paper) depends on the practical physical conditions (such as the initial condition and the boundary condition) of the cosmos, if we apply the solutions to the dynamical equations of the cosmic evolution. Thus, the effective cosmological constant value Λ is related closely to the cosmic boundary condition or the large-scale structure, which has a characteristic scale L_{U}. Thus, as a tentative analysis, the integration constant Λ should have the order of magnitude of L_{U}^{2} (for example, in the literature, Hoyle *et al* in their steady-state model for an expanding universe obtained a cosmological constant that equals $3H^{2}/c^{2}$ [31], where H denotes the Hubble constant. As is known, $3H^{2}/c^{2}$ has the same order of magnitude as L_{U}^{2}). This is the reason for why the observed cosmological constant has a nonzero but small value, which is close to the critical density of the universe.

Thus, we have analysed and explicated the cosmological constant problem (including some mysteries in connection with the quantum vacuum energy) by means of the spin-connection gravitational gauge field theory. Although the present scheme for revealing the physical origin and essence of the cosmological constant based on the spin-connection gauge field theory needs to be further investigated, it would be promising to address the cosmological constant problem, at least to provide a way to look at the anomalous characteristics of the cosmological constant from new aspects.

7. Concluding remarks

The affine connection and the fundamental dynamical variable in the Yang–Mills gauge theory are the same quantity. In contrast, the connection (Levi–Civita connection) and the dynamical variable (metric) in GR are not the same quantity. Thus, though GR is a kind of gauge theory whose gauge group consists of four diffeomorphisms and, in the vierbein formulation, of local frame rotations, yet it is not a Yang–Mills type gauge theory. It is possible, in fact, to demonstrate that this asymmetry between GR and Yang–Mills gauge field theory can be avoided if we introduce the formulation of *vielbein* (i.e., *vierbein* in four-dimensional spacetime. The 'vielbein' in four dimensions is referred to as the 'vierbein') for gravity (e.g., in the formulation of vierbein, where the spin connection is written in terms of the vierbein fields), one can identify the spin connection with the dynamical variable of a non-Abelian gauge field with a local Lorentz-group symmetry, and GR can then be reformulated as a gauge field theory of Yang–Mills type, where the gravitational Lagrangian (and hence the field equation) can be constructed based on the local Lorentz-group gauge invariance with the spin connection involved in the Yang–Mills covariant derivatives (in the vierbein formulation). In order to construct a renormalizable gravity theory, to interpret why the Newtonian gravitational constant has a dimension (or to interpret the physical origin of the Newtonian gravitational constant), as well as to unify the gravitational field with the Yang–Mills field, we suggest such a gravitational gauge field theory. We have shown that the Einstein equation of general relativity is in fact one of the first-integral solutions to the field equation of spin-connection Yang–Mills type gauge field. But it should be noted that in the conventional Yang–Mills gauge field theory, there was no 'vielbein' defined. As the spin connection in the gravitational gauge theory can be expressed in terms of the vierbein, one can, by analogy, conclude that the so-called 'Yang–Mills vielbein' could also be defined, and the Yang–Mills connection can be expressed in terms of this 'Yang–Mills vielbein' (i.e., the Yang–Mills gauge interaction can be reformulated in the formulation of 'Yang–Mills vielbein'). The present vielbein formulation of the Yang–Mills gauge field would be published elsewhere.

It is interesting that an 'equivalent cosmological constant' that is actually an integration constant appears in the first-integral solution to the spin-connection gravitational field equation. It naturally plays a mathematically equivalent role of the cosmological constant. In other words, our gauge field equation of Yang–Mills type can automatically exhibit such an effective cosmological constant, though the quantum vacuum energy no longer makes any contributions to the gravity. Obviously, the physical meaning of the present equivalent (effective) cosmological constant is no longer the density of vacuum energy or dark energy. Additionally, the idea that we view the cosmological constant as an integration constant may naturally interpret the observed cosmological constant value (close to the critical density) and would offer a solution to the cosmological constant problem, since the integration constant depends on the realistic physical conditions (such as the boundary conditions, the initial conditions and the large-scale structure of the gravitating system itself). To the best of our knowledge, there are at present no mechanisms for treating the cosmological constant problem based on the spin-connection gauge theory. Though we have provided new insights into the physical origin and meanings of the cosmological constant, the mechanism for interpreting the cosmological constant problem deserves further studies, since the present version is preliminary and tentative, and needs more improvements.

For the present, the curvature-only gravity theory has been tested experimentally and accepted as a standard gravity theory. However, the torsion-induced gravity theory deserves consideration, at least for the theoretical aspects (e.g., recently, the gravity with torsion receives increasingly more attentions from physicists [2, 32–35], and the possible detections of the torsion-based quantum interference and the torsion-induced gravitational interaction for the test particle spin have been suggested [36–38]). It seems desirable to include a Yang–Mills gauge field treatment for the gravity with torsion, i.e., the spin-connection gravitational gauge theory needs to be generalized to the case of torsion. We hope that the work presented here may stimulate an interest in this area, and would open a research route for looking at the cosmological constant problem, the spin contribution to gravitation [39] as well as the spin–torsion interaction [32].

Appendix. The generalization of the Lagrangian density of the φ–ϕ coupling

We have constructed a Lagrangian density in equation (21) for the φ–ϕ coupling as $\mathcal{L}_{\varphi-\phi} = \xi(\partial_\mu\phi^*\partial^\mu\varphi - m^2\phi^*\varphi) + \xi^*(\partial_\mu\phi\partial^\mu\varphi^* - m^2\phi\varphi^*)$, where the coupling parameters contain the mass square m^2 of the complex scalar fields. This Lagrangian density can, however, be generalized to the following form:

$$\mathcal{L}_{\varphi-\phi} = \xi(\partial_\mu\phi^*\partial^\mu\varphi - \mu^2\phi^*\varphi) + \xi^*(\partial_\mu\phi\partial^\mu\varphi^* - \mu^2\phi\varphi^*), \tag{A.1}$$

where we have replaced the special parameter m^2 with an arbitrary μ^2. Now this Lagrangian density can be rewritten as $\mathcal{L}_{\varphi-\phi} = \xi\phi^* J + \xi^*\phi J^* + \text{S.T.}$, where $J = -(\nabla_\mu\partial^\mu\varphi + \mu^2\varphi)$ and $J^* = -(\nabla_\mu\partial^\mu\varphi^* + \mu^2\varphi^*)$.

For the low-energy gravitational interaction, we have defined the 'quasi energy–momentum tensors' of the matter fields: $\tau_\alpha{}^\nu = \pi_\alpha{}^\nu - g_\alpha{}^\nu\mathcal{L}_\varphi, \tau^\mu{}_\beta = \pi^\mu{}_\beta - g^\mu{}_\beta\mathcal{L}_\varphi$. But since now we have a generalized $\mathcal{L}_{\varphi-\phi}$ in (A.1), we can use $\tau_\alpha{}^\nu = \pi_\alpha{}^\nu - g_\alpha{}^\nu\mathcal{L}_{\text{corr}}[\varphi^*,\varphi], \tau^\mu{}_\beta = \pi^\mu{}_\beta - g^\mu{}_\beta\mathcal{L}_{\text{corr}}[\varphi^*,\varphi]$ instead to represent the 'quasi energy–momentum tensors' of the matter fields, where $\mathcal{L}_{\text{corr}}[\varphi^*,\varphi] = \partial_\mu\varphi^*\partial^\mu\varphi - m^2_{\text{corr}}\varphi^*\varphi$. The corrected m^2_{corr} can be derived as follows: by the aid of $\mathcal{L}_\varphi = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi$ and $\mathcal{L}_{\text{int}} = \left(\xi^*\xi/m_G^2\right)(\nabla_\mu\partial^\mu\varphi^* + \mu^2\varphi^*)(\nabla_\nu\partial^\nu\varphi + \mu^2\varphi)$, one can obtain the equations of motion of the matter fields φ^*, φ (in the low-energy field theory of gravitation), e.g., $\nabla_\mu\partial^\mu\varphi + m^2\varphi = \left(\xi^*\xi/m_G^2\right)(\nabla_\mu\partial^\mu + \mu^2)(\nabla_\nu\partial^\nu\varphi + \mu^2\varphi)$. We assume that $\nabla_\mu\partial^\mu\varphi + m^2_{\text{corr}}\varphi = 0$, which can be rewritten as $\nabla_\mu\partial^\mu\varphi + m^2\varphi = \left(m^2 - m^2_{\text{corr}}\right)\varphi$ and $\nabla_\mu\partial^\mu\varphi + \mu^2\varphi = \left(\mu^2 - m^2_{\text{corr}}\right)\varphi$. Substitution of these two equations into the above equation of motion of the matter fields yields $m^2 - m^2_{\text{corr}} = \left(\xi^*\xi/m_G^2\right)\left(\mu^2 - m^2_{\text{corr}}\right)^2$. This can lead to the mass split of the matter fields φ^*, φ

$$m^2_{\text{corr}} = \mu^2 - \frac{m_G^2}{2\xi^*\xi} \pm \sqrt{\left(\frac{m_G^2}{2\xi^*\xi}\right)^2 + (m^2 - \mu^2)\frac{m_G^2}{\xi^*\xi}}. \tag{A.2}$$

Obviously, one of the roots, m_{corr}, would be reduced to m, if $\mu \to m$. This corresponds to the special case that has already been presented in the paper.

The covariant divergence of the present quasi energy–momentum tensor $\tau^{\mu\nu} = \partial^\mu\varphi^*\partial^\nu\varphi + \partial^\mu\varphi\partial^\nu\varphi^* - g^{\mu\nu}\mathcal{L}_{\text{corr}}[\varphi^*,\varphi]$ for the matter fields φ^*, φ is given by

$$\nabla_\mu\tau^{\mu\nu} = \left(\nabla_\mu\partial^\mu\varphi^* + m^2_{\text{corr}}\varphi^*\right)\partial^\nu\varphi + \left(\nabla_\mu\partial^\mu\varphi + m^2_{\text{corr}}\varphi\right)\partial^\nu\varphi^* = 0. \tag{A.3}$$

Hence, the terms $\nabla_\mu\tau^\mu{}_\beta$ and $\nabla_\mu\tau^\mu{}_\alpha$ in expression (40) vanish. This result plays a key role for obtaining equation (41) and the Einstein field equation (there would be no such a first-integral solution if $\nabla_\mu\tau^\mu{}_\beta$ and $\nabla_\mu\tau^\mu{}_\alpha$ are nonzero).

Now we turn to the classical field equations of both the matter fields and the heavy coupling field in high-energy processes. According to the Lagrangian density $\mathcal{L}_\varphi$, $\mathcal{L}_\phi$, and the new $\mathcal{L}_{\varphi-\phi}$ presented in (A.1), one can arrive at the following field equations of both φ and ϕ in the case of high-energy interaction: $\nabla_\mu\partial^\mu\varphi + m^2\varphi = -\xi^*(\nabla_\mu\partial^\mu\phi + \mu^2\phi)$, $\nabla_\mu\partial^\mu\phi + m_G^2\phi =$

$-\xi(\nabla_\mu \partial^\mu \varphi + \mu^2 \varphi)$. It follows that the field equations of ϕ and φ agree with

$$\begin{aligned} \nabla_\mu \partial^\mu \phi + \left(\frac{m_G^2 - \xi^* \xi \mu^2}{1 - \xi^* \xi}\right) \phi &= -\frac{\xi}{1 - \xi^* \xi} (\mu^2 - m^2) \varphi, \\ \nabla_\mu \partial^\mu \varphi + \left(\frac{m^2 - \xi^* \xi \mu^2}{1 - \xi^* \xi}\right) \varphi &= -\frac{\xi^*}{1 - \xi^* \xi} \left(\mu^2 - m_G^2\right) \phi. \end{aligned} \tag{A.4}$$

The first equation (of ϕ field) would be reduced to equation (42), if the parameter $\mu \to m$.

In a word, the φ–ϕ coupling that has been considered in the preceding sections is in fact a special case of the one presented in appendix.

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